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Current algebras over manifolds: Poisson algebras, q-deformations and quantization

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Abstract

Poisson algebras on current manifolds (of maps from a finite-dimensional manifold into a two-dimensional manifold) are investigated using symplectic geometry. It is shown that there is a one-to-one correspondence between such current manifolds and Poisson current algebras with three generators. A geometric meaning is given to q-deformations of current algebras. The geometric quantization of current algebras and quantum current algebraic maps is also studied. © 1998 Elsevier Science B.V.

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1. Introduction

Current algebras were first studied in particle physics [1]. The primary ingredients of current algebras are the sets of equal-time commutation relations for the physically conserved currents [2]. Mathematically, current algebras are maps from a (compact) manifold N to an algebra g. When N is the one-dimensional manifold S^1, current algebras are usually called loop algebras (see e.g. [3–6]). The representation theory of current algebras has been studied quite extensively, see e.g. [3–5]. In connection with the theory of integrable systems certain infinite-dimensional "classical Poisson algebras" have been investigated, see e.g. [7].

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In the present paper we study certain current algebras associated with maps from a finite Riemannian manifold into a two-dimensional (2D) Riemannian manifold (some of the considerations however are independent of the chosen Riemannian structures). We call these manifolds "2D current manifolds". By investigating the symplectic geometry on these 2D current manifolds, we show that there is a one-to-one correspondence between 2D current manifolds and Poisson current algebras with three generators, including current Lie algebras and q-deformed current Lie algebras, which gives an infinite-dimensional extension of the corresponding results in the finite-dimensional case [8,9]. By geometric quantization, we get corresponding quantized current algebras and related current manifolds in quantum version. Both the Poisson current algebraic maps (resp. quantum current algebraic maps) can then be investigated in terms of the corresponding classical (resp. quantum current) manifolds.

We first recall in Section 2 some notations of infinite-dimensional symplectic geometry referring to [10,11] for background. In Section 3 we study the symplectic geometry on 2D current manifolds and establish relations between Poisson current structures and 2D current manifolds. As applications, we discuss in Section 4 some special 2D current manifolds and their related Poisson current manifolds. Section 5 is dedicated to Poisson current algebraic maps in terms of the corresponding current manifolds. We investigate the geometric quantization of current algebras in Section 6 and give some concluding remarks in Section 7.

2. Symplectic geometry on current manifolds

The basic object in symplectic geometry is a symplectic manifold which is an even-dimensional manifold equipped with a symplectic two-form, see e.g. [11,12]. Let M be a connected even-dimensional Riemannian manifold and N an arbitrary finite-dimensional Riemannian manifold equipped with a finite reference measure μ. A current manifold M_N is the space of smooth mappings from N to M, which can be equipped with the topology of a Banach manifold, see e.g. [4]. Let δ denote the exterior derivative on M. By definition a symplectic form ω on M_N is a two-form on M with parameters in N, which is: (i) closed: $\delta\omega = 0$ and (ii) non-degenerate: $X\rfloor\omega = 0 \Rightarrow X = 0$, where X are vector fields on M_N and $\rfloor$ denotes the left inner product defined by $(X\rfloor\omega)(Y) = \omega(X, Y)$ for any two smooth vector fields X and Y on M_N. It is possible to show that such symplectic forms exist on M_N, see e.g. [10,11] (for the case $\dim(M) = 2$ we discuss this below).

Canonical transformations are by definition ω-preserving diffeomorphisms of M_N onto itself. A vector X on M_N corresponds to an infinitesimal canonical transformation if and only if the Lie derivative of ω with respect to X vanishes, $\mathcal{L}_X\omega = X\rfloor\delta\omega + \delta(X\rfloor\omega) = 0$. Such a vector X is said to be a Hamiltonian vector field. Since ω is closed, it follows that a vector X is a Hamiltonian vector field if and only if $X\rfloor\delta$ is closed. Let $\mathcal{F}(M_N)$ denote the real-valued smooth functions on M_N. For $f \in \mathcal{F}(M_N)$, since ω is non-degenerate there exists a Hamiltonian vector field X_f (unique up to a sign on the right-hand side of the following equation) satisfying

$$X_f \rfloor \omega = -\delta f. \tag{1}$$

The Poisson bracket $[f, g]_{\rm PB}$ of two smooth functions f and g in $\mathcal{F}(M_N)$ is defined to be the function $-\omega(X_f, X_g)$. It satisfies the identities:

$$[f, g]_{\rm PB} = -\omega(X_f, X_g) = \omega(X_g, X_f) = -X_f g = X_g f. \tag{2}$$

According to Whitney's embedding theorems we can smoothly embed N (resp. M) in Euclidean spaces of dimensions $n(\leq 2(\dim N + 1))$ (resp. $m(\leq 2(\dim M + 1))$), see e.g. [13]. Let $\mathbf{x}$ (resp. S) be local coordinates of the so embedded manifold N (resp. M). Let $S_i, i = 1, \ldots, m$, be the components of S. The basis of the tangent vectors (resp. cotangent vectors) of the Banach manifold M_N are then $\{\delta/\delta S_i(\mathbf{x})\}$ (resp. $\{\delta S_i(\mathbf{x})\}$), $i = 1, \ldots, m$. For fixed $\mathbf{x}$, $S_1(\mathbf{x}), \ldots, S_m(\mathbf{x})$ can be looked upon as orthogonal smooth vectors spanning the tangent space at $\mathbf{x} \in N$ to M. In analogy with the finite-dimensional situation (see e.g. [11]) we can define an innner product between the bases of tangent vectors on M_N and two forms of M_N

$$\begin{aligned} &\frac{\delta}{\delta S_i(\mathbf{x})} \rfloor \delta S_j(\mathbf{y}) \wedge \delta S_k(\mathbf{z}) \\ &\quad = \delta_{ij}\delta(\mathbf{x}-\mathbf{y})\delta S_k(\mathbf{z}) - \delta_{ik}\delta(\mathbf{x}-\mathbf{z})\delta S_j(\mathbf{y}), \quad i, j, k = 1, \ldots, m \end{aligned}$$

(where $\delta(\cdot)$ is the usual δ function on N and $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$). This equality is to be understood in the sense of generalized functions (using the natural pairing given by the Riemann–Lebesgue volume measures on N and M).

3. Poisson current algebraic structures on 2D current manifolds

In the following we take M to be a two-dimensional Riemannian manifold smoothly embedded into $\mathbb{R}^3$. Let, as in Section 2, $S_i, i = 1, 2, 3$, be the coordinates of M in $\mathbb{R}^3$ and $\mathbf{x}$ the coordinate vector of the manifold N in $\mathbb{R}^n$ (n is as in Section 2). We consider a general "2D current manifold" M_N defined in terms of some smooth real-valued function F on $\mathbb{R}^3$ by

$$F(\overset{\circ}{S}(\mathbf{x})) = 0, \quad \mathbf{x} \in N, \tag{3}$$

$\overset{\circ}{S}$ denoting the values of $S = (S_1, S_2, S_3)$ for which (3) holds. The Poisson algebraic structure on the current manifold (3) is determined by the corresponding symplectic structure on it.

Let for general $S \equiv (S_i, i = 1, 2, 3) \in M_N$

$$\mathbb{F}(S) \equiv \int_N F(S)(\mathbf{x}))\, \mathrm{d}\mu(\mathbf{x}),$$

where μ is the Riemann–Lebesgue volume measure of the Riemannian manifold N. We look at $\mathbb{F}$ as a real-valued smooth functional of $S = (S_i, i = 1, 2, 3)$. When $S = \overset{\circ}{S}$ then

(3) holds, hence $\mathbb{F}(S) = 0$. We define $\delta\mathbb{F}/\delta S_j(\mathbf{x})$ as the smooth functional of S s.t. for $h_j \in C(N)$

$$\int h_j(\mathbf{x}) \frac{\delta\mathbb{F}(S)}{\delta S_j(\mathbf{x})} \,\mathrm{d}\mu(\mathbf{x}) \equiv \delta\mathbb{F}(S; h_j) \equiv \lim_{\epsilon\downarrow 0} \frac{\mathbb{F}(S^{\epsilon h_j}) - \mathbb{F}(S)}{\epsilon}$$

with $S^{\epsilon h_j} = (\ldots, S_j + \epsilon h_j, \ldots)$, $j = 1, 2, 3$. $\delta\mathbb{F}(S; h_j)$ is thus the derivative of $\mathbb{F}$ at S in the direction h_j. We assume that N is compact (or that μ and the functions to be integrated against it which occur in our formulae are such that all integrals are finite).

The Hamiltonian vector fields $X_{S_i(\mathbf{x})}$, resp. the symplectic form, on M_N have the following general forms:

$$X_{S_i(\mathbf{x})} = \sum_{jk=1}^{3} \epsilon_{ijk} A_j(\mathbf{x}) \frac{\delta}{\delta S_k(\mathbf{x})}, \tag{4}$$

resp.

$$\omega = -\frac{1}{2} \sum_{lmn=1}^{3} \int_N \epsilon_{lmn} B_l(\mathbf{y}) \delta S_m(\mathbf{y}) \wedge \delta S_n(\mathbf{y}) \,\mathrm{d}\mu(\mathbf{y}), \tag{5}$$

where $A_i(\mathbf{x})$ and $B_i(\mathbf{x})$, $i = 1, 2, 3$, are some smooth functions of $S(\mathbf{x})$ satisfying Eq. (1). Their determination is given in the following proposition.

Proposition 1. *The Hamiltonian vector fields associated with $S = (S_i(\mathbf{x}), i = 1, 2, 3) \in M_N$ are more precisely given by*

$$X_{S_i(\mathbf{x})} = \alpha \sum_{jk=1}^{3} \epsilon_{ijk} \frac{\delta\mathbb{F}(S)}{\delta S_j(\mathbf{x})} \frac{\delta}{\delta S_k(\mathbf{x})}, \tag{6}$$

where α is a real constant and ϵ_{ijk} is the completely antisymmetric tensor.

Proof. By definition a Hamiltonian vector field X_f associated with $f \in \mathcal{F}(M_N)$ should satisfy Eq. (1). Substituting (4) and (5) into Eq. (1) we have (in the sense of generalized functions)

$$\begin{aligned} X_{S_i(\mathbf{x})} \rfloor \omega &= -\frac{1}{2} \sum_{jklmn=1}^{3} \int_n \epsilon_{ijk}\epsilon_{lmn} A_j(\mathbf{x}) B_l(\mathbf{y}) \delta(\mathbf{x}-\mathbf{y}) \\ &\quad \times [\delta_{km} \delta S_n(\mathbf{y}) - \delta_{kn} \delta S_m(\mathbf{y})] \,\mathrm{d}\mu(\mathbf{y}) \\ &= -\frac{1}{2} \sum_{jlmn=1}^{3} \epsilon_{lmn} A_j(\mathbf{x}) B_l(\mathbf{x}) [\epsilon_{ijm} \delta S_n(\mathbf{x}) - \epsilon_{ijn} \delta S_m(\mathbf{x})] \\ &= -\sum_{jlmn=1}^{3} \epsilon_{ijm}\epsilon_{lmn} A_j(\mathbf{x}) B_l(\mathbf{x}) \delta S_n(\mathbf{x}) \\ &= -\delta S_i(\mathbf{x}), \quad i = 1, 2, 3. \end{aligned} \tag{7}$$

For $i = 1$ we have $X_{S_1(\mathbf{x})} \rfloor \omega = -\delta S_1(\mathbf{x})$, i.e.,

$$-(A_2)(\mathbf{x})B_2(\mathbf{x}) + A_3(\mathbf{x})B_3(\mathbf{x}))\delta S_1(\mathbf{x}) + A_2(\mathbf{x})B_1(\mathbf{x})\delta S_2(\mathbf{x}) \\ + A_3(\mathbf{x})B_1(\mathbf{x})\delta S_3(\mathbf{x}) = -\delta S_1(\mathbf{x}).$$

$\delta S_i(\mathbf{x})$, $i = 1, 2, 3$, are not linearly independent, in fact from Eq. (3) we have

$$\sum_{i=1}^{3} \frac{\delta \mathbb{F}(S)}{\delta S_i} \delta S_i \bigg|_N = 0. \tag{8}$$

Using relation (8) we get

$$A_2(\mathbf{x}) = \alpha \frac{\delta \mathbb{F}(S)}{\delta S_2(\mathbf{x})}, \qquad A_3(\mathbf{x}) = \alpha \frac{\delta \mathbb{F}(S)}{\delta S_3(\mathbf{x})}$$

for some real constant α and

$$\alpha \left(B_1(\mathbf{x}) \frac{\delta \mathbb{F}(S)}{\delta S_1(\mathbf{x})} + B_2(\mathbf{x}) \frac{\delta \mathbb{F}(S)}{\delta S_2(\mathbf{x})} + B_3(\mathbf{x}) \frac{\delta \mathbb{F}(S)}{\delta S_3(\mathbf{x})} \right) = 1.$$

Combining this with the corresponding results from (7) with $i = 2, 3$ we obtain

$$A_i(\mathbf{x}) = \alpha \frac{\delta \mathbb{F}(S)}{\delta S_i(\mathbf{x})}, \quad i = 1, 2, 3 \tag{9}$$

and

$$\alpha \sum_{i=1}^{3} B_i(\mathbf{x}) \frac{\delta \mathbb{F}(S)}{\delta S_i(\mathbf{x})} = 1. \tag{10}$$

Substituting Eq. (9) into (4) we get (6). □

Remark 1. It is easily seen that $A_i(\mathbf{x})$ is independent of the coefficients $B_i(\mathbf{x})$, i.e., the Hamiltonian vector field associated with $S_i(\mathbf{x})$ is independent of the construction of the symplectic form ω on M_N. Owing to the equivalence of Hamiltonian vector fields in the 2D case [8], for simplicity the factor α will be taken from now on to be $1/2$ (different values of α give rise to the same algebra up to an algebraic isomorphism).

Proposition 2. *The two-form ω given by (5) is a symplectic form on M_N iff $B_i(\mathbf{x})$, $i = 1, 2, 3$, satisfy condition (10).*

Proof. It is manifest from the proof above that condition (10) is necessary and sufficient for ω to satisfy formula (1). Further, due to the fact that M is a 2D manifold, ω is obviously closed, i.e., $\delta\omega = 0$. □

On finite-dimensional manifolds with a symplectic structure one can define Poisson algebraic structures. In a similar way we define the "Poisson current algebraic structures" on current manifolds equipped with a symplectic structure.

Theorem 1. *The Poisson current algebraic structure on the manifold M_N is (uniquely) given by*

$$[S_i(\mathbf{x}), S_j(\mathbf{y})]_{\mathrm{PB}} = \frac{1}{2}\sum_{k=1}^{3} \epsilon_{ijk} \frac{\delta \mathbb{F}(S)}{\delta S_k(\mathbf{x})} \delta(\mathbf{x}-\mathbf{y}), \tag{11}$$

the equality being in the sense of generalized functions over N.

Proof. From formula (2) and Proposition 1 (with $\alpha = 1/2$) we have

$$\begin{aligned}[S_i(\mathbf{x}), S_j(\mathbf{y})]_{\mathrm{PB}} &= -X_{S_{i(\mathbf{x})}} S_j(\mathbf{y}) \\ &= \frac{1}{2}\sum_{k=1}^{3} \epsilon_{ijk} \frac{\delta \mathbb{F}(S)}{\delta S_k(\mathbf{x})} \delta(\mathbf{x}-\mathbf{y}). \qquad \square\end{aligned}$$

This Poisson current algebra is independent of the symplectic form on M_N, and is uniquely given by the current manifold $\mathbb{F}(S) = 0$ under the algebraic equivalence discussed in [8].

Proposition 3. *For $f \in \mathcal{F}(M_N)$, the Hamiltonian vectorfield associated with f is given by*

$$X_f = \frac{1}{2}\int_N \sum_{ijk=1}^{3} \epsilon_{ijk} \frac{\delta f}{\delta S_i(\mathbf{y})} \frac{\delta \mathbb{F}(S)}{\delta S_j(\mathbf{y})} \frac{\delta}{\delta S_k(\mathbf{y})}\, \mathrm{d}\mu(\mathbf{y}). \tag{12}$$

Proof. We have to prove $X_f \rfloor \omega = -\delta f$. From

$$-\delta f = -\int_N \sum_{i=1}^{3} \frac{\delta f}{\delta S_i(\mathbf{x})} \delta S_i(\mathbf{x})\, \mathrm{d}\mu(\mathbf{x}),$$

and Proposition 1 we have

$$-\delta S_i(\mathbf{x}) = X_{S_{i(\mathbf{x})}} \rfloor \omega = \frac{1}{2}\sum_{jk=1}^{2} \epsilon_{ijk} \frac{\delta \mathbb{F}(S)}{\delta S_j(\mathbf{x})} \frac{\delta}{\delta S_k(\mathbf{x})} \rfloor \omega.$$

Therefore

$$-\delta f = \frac{1}{2}\int_N \sum_{ijk=1}^{3} \epsilon_{ijk} \frac{\delta f}{\delta S_i(\mathbf{y})} \frac{\delta \mathbb{F}(S)}{\delta S_j(\mathbf{y})} \frac{\delta}{\delta S_k(\mathbf{y})}\, \mathrm{d}\mu(\mathbf{y}) \rfloor \omega.$$

Comparing above formula with the condition $X_f \rfloor \omega = -\delta f$ we get formula (12). $\square$

Theorem 2. *For $f, g \in \mathcal{F}(M_N)$, the Poisson bracket of f and g is given by*

$$[f, g]_{\mathrm{PB}} = -\frac{1}{2}\int_N \sum_{ijk}^{3} \epsilon_{ijk} \frac{\delta f}{\delta S_i(\mathbf{y})} \frac{\delta \mathbb{F}(S)}{\delta S_j(\mathbf{y})} \frac{\delta g}{\delta S_k(\mathbf{y})}\, \mathrm{d}\mu(\mathbf{y}). \tag{13}$$

Proof. This is a direct result of Proposition 3 and formula (2).

Generally, Poisson current algebras are by definition maps from the compact manifold N to finite-dimensional algebras, equipped with a Poisson bracket. Let us consider a general Poisson current algebra

$$[S_i(\mathbf{x}), S_j(\mathbf{y})]_{\mathrm{PB}} = \sum_{k=1}^{3} \epsilon_{ijk} f_k(S(\mathbf{x}))\delta(\mathbf{x}-\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in N, \quad S \in \mathbb{R}^3 \tag{14}$$

(in the sense of distribution on N), where f_i, $i = 1, 2, 3$, are smooth functions of S taking values on a finite-dimensional manifold N and $S_i(\mathbf{x})$, $i = 1, 2, 3$, satisfy the Jacobi identity.

Definition. *If the f_i, $i = 1, 2, 3$, satisfy*

$$\left.\frac{\partial f_i}{\partial S_j}\right|_p = \left.\frac{\partial f_j}{\partial S_i}\right|_p, \quad p \in N, \quad i, j = 1, 2, 3, \tag{15}$$

then the algebra (14) is said to be integrable.

Remark 2. The "integrability condition" (15) is a sufficient condition for a general Poisson current algebra (14) to satisfy the Jacobi identity,

$$\begin{aligned}
&[S_1(\mathbf{x}), [S_2(\mathbf{y}), S_3(\mathbf{z})]_{\mathrm{PB}}]_{\mathrm{PB}} + [S_2(\mathbf{y}), [S_3(\mathbf{z}), S_1(\mathbf{x})]_{\mathrm{PB}}]_{\mathrm{PB}} \\
&\quad + [S_3(\mathbf{z}), [S_1(\mathbf{x}), S_2(\mathbf{y})]_{\mathrm{PB}}]_{\mathrm{PB}} \\
&\quad = \left.\left[\frac{\partial f_1}{\partial S_2} f_3 - \frac{\partial f_1}{\partial S_3} f_2 + \frac{\partial f_2}{\partial S_3} f_1 - \frac{\partial f_2}{\partial S_1} f_3 + \frac{\partial f_3}{\partial S_1} f_2 - \frac{\partial f_3}{\partial S_2} f_1\right]\right|_{\mathbf{x}} \\
&\quad \delta(\mathbf{x}-\mathbf{y})\delta(\mathbf{y}-\mathbf{z}) = 0.
\end{aligned}$$

Remark 3. Comparing formula (11) with formula (14) we see that the $f_i(S(\mathbf{x}))$ in the Poisson current bracket (11) on M_N is given by

$$f_i(S(\mathbf{x})) = \frac{1}{2}\frac{\delta \mathbb{F}(S)}{\delta S_i(\mathbf{x})}, \tag{16}$$

where $\mathbb{F}(S) = \int_N F(S(\mathbf{x}))\,\mathrm{d}\mu(\mathbf{x})$. As $F \in \mathcal{F}(M_N)$ we have

$$\left.\frac{\partial^2 F(S)}{\partial S_i \partial S_j}\right|_p = \left.\frac{\partial^2 F(S)}{\partial S_j \partial S_i}\right|_p, \quad p \in N, \quad i, j, = 1, 2, 3.$$

Therefore the f_i in (16) satisfy the integrability condition (15) and all the Poisson current algebras (11) in Theorem 1, over the current manifold M_N, considered in this section are integrable.

Theorem 3. *Let M and N be Riemannian manifolds smoothly embedded in $\mathbb{R}^3$ and $\mathbb{R}^n$, respectively. For a given integrable current Poisson algebra (14), there exists a symplectic current manifold M_N described by an equation of the form $\int_N F(S(\mathbf{y}))\,\mathrm{d}\mathbf{y} = c$, with $S(\mathbf{y}) \in M$, $\mathbf{y} \in N$, $F \in \mathcal{F}(M_N)$ and c an arbitrary real number, such that the Poisson current algebra generated by $\{S(\mathbf{y}), \mathbf{y} \in N, S \in M_N\}$ coincides with the algebra* (14).

Proof. A general integrable Poisson algebra is of the form (14), with f_i, $i = 1, 2, 3$, satisfying the integrability condition (15). What we have to show is that this Poisson algebra can be described by the symplectic geometry on a suitable symplectic current manifold (M_N, ω), in the sense that the above Poisson current bracket can be described by formula (2), i.e., the Poisson current bracket $[S_i(\mathbf{x}), S_j(\mathbf{y})]_{\rm PB}$ is given by the Hamiltonian vector field $X_{S_i(\mathbf{x})}$ associated with $S_i(\mathbf{x})$ such that

$$[S_i(\mathbf{x}), S_j(\mathbf{y})]_{\rm PB} = -X_{S_i(\mathbf{x})} S_j(\mathbf{y}) = \sum_{k=1}^{3} \epsilon_{ijk} f_k(S(\mathbf{x}))\delta(\mathbf{x}-\mathbf{y}). \tag{17}$$

Let $X'_{S_i(\mathbf{x})}$ be given by

$$X'_{S_i(\mathbf{x})} \equiv \sum_{jk=1}^{3} \epsilon_{ijk} f_j(\mathbf{x}) \frac{\delta}{\delta S_k(\mathbf{x})}. \tag{18}$$

Then $X'_{S_i(\mathbf{x})}$ satisfies (17) with X replaced by X'.

A general two-form ω' on $\mathbb{R}^3_N$ has the form (5). We have to prove that $S \in \mathbb{R}^3_N$ can be restricted to a suitable two-dimensional manifold $M \subset \mathbb{R}^3$ in such a way that $X'_{S_i(\mathbf{x})}$ coincides with the Hamiltonian vector field $X_{S_i(\mathbf{x})}$ and ω' is the corresponding symplectic form ω on M_N.

A two-form on M_N is always closed. What we should then check is that there exists $M \subset \mathbb{R}^3$ such that for S restricted to M formula (1) holds for $f = S_i(\mathbf{x})$, i.e.,

$$X'_{S_i(\mathbf{x})} \rfloor \omega' = -\delta S_i(\mathbf{x}), \quad i = 1, 2, 3. \tag{19}$$

Substituting formulae (18) and (5) into (19) we get

$$X'_{S_i(\mathbf{x})} \rfloor \omega' = -\sum_{jlmn=1}^{3} \epsilon_{ijm}\epsilon_{lmn} f_j(\mathbf{x}) B_l(\mathbf{x}) \delta S_n(\mathbf{x}) = -\delta S_i(\mathbf{x}).$$

That is,

$$\begin{aligned}(1 - f_2(\mathbf{x})B_2(\mathbf{x}) - f_3(\mathbf{x})B_3(\mathbf{x}))\delta S_1(\mathbf{x}) + f_2(\mathbf{x})B_1(\mathbf{x})\delta S_2(\mathbf{x})\\ + f_3(\mathbf{x})B_1(\mathbf{x})\delta S_3(\mathbf{x}) = 0\end{aligned} \tag{20}$$

and cyclically.

Let us now look at the coefficient determinant D of the $\delta S_i(\mathbf{x})$ in the system (20). By a suitable choice of (B_1, B_2, B_3) we can obtain that D is zero. This is in fact equivalent with the equation

$$f_1(\mathbf{x})B_1(\mathbf{x}) + f_2(\mathbf{x})B_2(\mathbf{x}) + f_3(\mathbf{x})B_3(\mathbf{x}) = 1 \tag{21}$$

being satisfied. The fact that $D = 0$ implies that there exists indeed an M as above.

Substituting condition (21) into (20) we get

$$f_1(\mathbf{x})\delta S_1(\mathbf{x}) + f_2(\mathbf{x})\delta S_2(\mathbf{x}) + f_3(\mathbf{x})\delta S_3(\mathbf{x}) = 0. \tag{22}$$

From assumption (15) we know that the differential equation (22) is exactly solvable, in the sense that there exists a smooth (potential) function $F \in \mathcal{F}(M_N)$ and a constant c such that

$$\mathbb{F}(S) \equiv \int_N F(S(\mathbf{y}))\,\mathrm{d}\mathbf{y} = c \tag{23}$$

and $\delta\mathbb{F}/\delta S_i(\mathbf{x}) = f_i(S(\mathbf{x}))$. The above manifold M_N is then described by (23).

Therefore for any given integrable Poisson current algebra there always exists a current manifold of the form (23) on which $X'_{S_i(\mathbf{x})}$ in (18) is a Hamiltonian vector field and the Poisson bracket of the current algebra is given by $X'_{S_i(\mathbf{x})}$ according to the formula (17).

The current manifold defined by (23) is unique (once c is given). Hence an integrable Poisson current algebra is uniquely given by a current manifold M_N described by $\mathbb{F}(S) \equiv \int_N F(S(\mathbf{y}))\,\mathrm{d}\mathbf{y} = c$ for some F and c. □

4. Poisson algebraic structures on some special current manifolds

In this section we discuss Poisson algebraic structures on some special current manifolds, which give rise to special current extensions of Poisson–Lie algebras and q-deformed Lie algebras. In all examples below $\mathbf{x}$ takes values in a Riemannian manifold N, smoothly embedded in $\mathbb{R}^n$.

(a) We first consider a "current 2D sphere" given by

$$S_1^2(\mathbf{x}) + S_2^2(\mathbf{x}) + S_3^2(\mathbf{x}) = S_0^2, \tag{24}$$

where S_0 is a real constant $\neq 0$.

M is then here the sphere in $\mathbb{R}^3$ of radius S_0. From Theorem 1 we have the Poisson relations (in the sense of distributions),

$$[S_i(\mathbf{x}), S_j(\mathbf{y})]_{\mathrm{PB}} = \sum_{k=1}^{3} \epsilon_{ijk} S_k(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) \tag{25}$$

The algebra defined by (25) is a current extension of the Poisson algebra $SU(2)$ (cf. see e.g. [1,2,5,14]).

A symplectic form on the current manifold M_N can be constructed by using formula (5) and condition (10). From (10) and (24) we have

$$B_1(\mathbf{x})S_1(\mathbf{x}) + B_2(\mathbf{x})S_2(\mathbf{x}) + B_3(\mathbf{x})S_3(\mathbf{x}) = 1.$$

Comparing this with Eq. (24) we can simply take $B_i(\mathbf{x}) = S_i(\mathbf{x})/S_0^2$. Therefore from (5) we obtain the following symplectic form:

$$\omega = \frac{-1}{2S_0^2} \sum_{ijk=1}^{3} \int_N \epsilon_{ijk} S_i(\mathbf{x})\delta S_j(\mathbf{x}) \wedge \delta S_k(\mathbf{x})\,\mathrm{d}\mu(\mathbf{x}). \tag{26}$$

(b) We now consider a "q-deformed current 2D sphere" defined by

$$S_1^2(\mathbf{x}) + S_2^2(\mathbf{x}) + \frac{(\sinh\gamma S_3(\mathbf{x}))^2}{\gamma\sinh\gamma} = \frac{(\sinh\gamma S_0)^2}{\gamma\sinh\gamma} \stackrel{\text{def}}{=} S_\gamma^2, \tag{27}$$

where $\gamma = \log q$, $q \in \mathbb{R}\backslash\{0\}$ is the deformation parameter. Heuristically, when $\gamma \to 0$, the manifold (27) becomes the current 2D sphere (24).

For the manifold (27), Theorem 1 gives rise to the following Poisson algebraic relations (again written in the sense of generalized functions):

$$\begin{aligned} [S_+(\mathbf{x}), S_-(\mathbf{y})]_{\mathrm{PB}} &= -\mathrm{i}\frac{\sinh 2\gamma S_3(\mathbf{x})}{\sinh\gamma}\delta(\mathbf{x}-\mathbf{y}), \\ [S_3(\mathbf{x}), S_\pm(\mathbf{y})]_{\mathrm{PB}} &= \mp\mathrm{i}S_\pm(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}), \end{aligned} \tag{28}$$

where $S_\pm(\mathbf{x}) = S_1(\mathbf{x}) \pm \mathrm{i}S_2(\mathbf{x})$ and $\mathrm{i} = \sqrt{-1}$.

The algebra (28) is just the current extension of the q-deformed Poisson–Lie algebra $SU_q(2)$ [15]. It is isomorphic (up to a factor i) to the current extension of the "quantum" algebra $SU_q(2)$, but is here classically realized, which means that the q-deformation and physical $\hbar$-quantization of the current extended algebras are independent, like in the case of Lie algebras [9,16]. Both current extended Lie algebras and current extended q-deformed Lie algebras can thus be realized at classical as well as at quantum levels (see Section 6).

The symplectic form of this example can be similarly obtained from condition (10) and Eq. (27):

$$\begin{aligned} \omega = -\frac{\gamma\sinh\gamma}{(\sinh\gamma S_0)^2}\int_N \Bigg[& S_1(\mathbf{x})\delta S_2(\mathbf{x}) \wedge \delta S_3(\mathbf{x}) \\ & + S_2(\mathbf{x})\delta S_3(\mathbf{x}) \wedge \delta S_1(\mathbf{x}) \\ & + \frac{\tanh\gamma S_3(\mathbf{x})}{\gamma}\delta S_1(\mathbf{x}) \wedge \delta S_2(\mathbf{x})\Bigg]\mathrm{d}\mu(\mathbf{x}). \end{aligned} \tag{29}$$

(c) The "current elliptic paraboloid" is defined by

$$S_1^2(\mathbf{x}) + S_2^2(\mathbf{x}) - S_3(\mathbf{x}) = \tfrac{1}{2}. \tag{30}$$

From formula (11) we have, in the sense of generalized functions:

$$\begin{aligned} [S_1(\mathbf{x}), S_2(\mathbf{y})]_{\mathrm{PB}} &= -\tfrac{1}{2}\delta(\mathbf{x}-\mathbf{y}), \\ [S_2(\mathbf{x}), S_3(\mathbf{y})]_{\mathrm{PB}} &= S_1(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}), \\ [S_3(\mathbf{x}), S_1(\mathbf{y})]_{\mathrm{PB}} &= S_2(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{31}$$

This is just the current extension of the Poisson simple harmonic oscillator algebra $\mathcal{H}$ (4) [17].

The corresponding symplectic form can be obtained by using formulae (5), (10) and Eq. (30):

$$\omega = -2 \int_N [S_1(\mathbf{x})\delta S_2(\mathbf{x}) \wedge \delta S_3(\mathbf{x}) + S_2(\mathbf{x})\delta S_3(\mathbf{x}) \wedge \delta S_1(\mathbf{x}) + 2S_3(\mathbf{x})\delta S_1(\mathbf{x}) \wedge \delta S_2(\mathbf{x})]\, \mathrm{d}\mu(\mathbf{x}). \tag{32}$$

(d) The "q-deformed current elliptic paraboloid" is defined by

$$S_1^2(\mathbf{x}) + S_2^2(\mathbf{x}) - \frac{\sinh(2\gamma S_3(\mathbf{x}))}{2\gamma \cosh\gamma} = \frac{\sinh\gamma}{2\gamma\cosh(\gamma)}, \tag{33}$$

where again $\gamma = \log q$, $q \in \mathbb{R}\backslash\{0\}$ is the deformation parameter.

The algebra on this current manifold is the q-deformed current extension of the simple harmonic oscillator algebra of $\mathcal{H}_q(4)$ considered in [18] and is described (in the sense of generalized functions) by:

$$\begin{aligned}
[S_1(\mathbf{x}), S_2(\mathbf{y})]_{\mathrm{PB}} &= -\frac{\cosh(2\gamma S_3(\mathbf{x}))}{2\cosh\gamma}\delta(\mathbf{x}-\mathbf{y}),\\
[S_2(\mathbf{x}), S_3(\mathbf{y})]_{\mathrm{PB}} &= S_1(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}),\\
[S_3(\mathbf{x}), S_1(\mathbf{y})]_{\mathrm{PB}} &= S_2(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}).
\end{aligned} \tag{34}$$

The symplectic form is given by

$$\begin{aligned}
\omega = -2\frac{\gamma\cosh\gamma}{\sinh\gamma}\int_N \Bigg[& S_1(\mathbf{x})\delta S_2(\mathbf{x}) \wedge \delta S_3(\mathbf{x})\\
& + S_2(\mathbf{x})\delta S_3(\mathbf{x}) \wedge \delta S_1(\mathbf{x})\\
& + \frac{\sinh(2\gamma S_3(\mathbf{x}))}{\gamma\cosh(2\gamma S_3(\mathbf{x}))}\delta S_1(\mathbf{x}) \wedge \delta S_2(\mathbf{x})\Bigg]\, \mathrm{d}\mu(\mathbf{x}).
\end{aligned} \tag{35}$$

Using formulae (11), (5) and (10), other examples of current extended Poisson algebras and related symplectic structures associated with current manifolds can be constructed in a similar way. For instance one may check that to the "current manifold of the one sheet hyperboloid" defined by $S_1^2(\mathbf{x}) + S_2^2(\mathbf{x}) - S_3^2(\mathbf{x}) =$ constant there belongs the current extension of the well known Poisson $SU(1,1)$ algebra.

5. Poisson current algebraic maps

From the above we see that the current algebras which are defined as maps from a compact manifold N to an algebra with three generators are related, via symplectic geometry, to certain current manifolds and vice versa. Therefore it is convenient to investigate current algebraic maps by using the associated current manifolds.

Let A_0 and A_0' be algebras with three generators. Let N and N' be (Riemannian) manifolds smoothly embedded in $\mathbb{R}^3$ such that $D \equiv N \cap N' \neq \emptyset$. Let A and A' be current algebras of mappings from N and N' to A_0 and A_0', respectively. Let $S \equiv S_i(\mathbf{x})$ (resp. $S' \equiv S_i'(\mathbf{x})$), $i = 1, 2, 3$, $\mathbf{x} \in D$, be the generators of the current algebra A (resp. A'), with corresponding current manifolds $F(S) = 0$ (resp. $F'(S') = 0$) defined by a certain smooth real-valued function F (resp. F').

Theorem 4. *$\tilde{S}(S) \equiv \{\tilde{S}_i(\mathbf{x}), i = 1, 2, 3; \mathbf{x} \in D\}$ generates A' iff $\tilde{S}$ satisfies $\mathbb{F}'(\tilde{S}) = 0$, where $\mathbb{F}'(\tilde{S}) \equiv \int_D F'(\tilde{S})\, \mathrm{d}\mu$ and $\mathbb{F}(S) \equiv \int_D F(S)\, \mathrm{d}\mu = 0$.*

Proof. Let M_D be the manifold defined by the equation $F(S(\mathbf{x})) = 0$, $\mathbf{x} \in D$ and M'_D the manifold defined by the equation $F'(S'(\mathbf{x})) = 0$, $\mathbf{x} \in D$. If $\tilde{S}$ satisfies $\mathbb{F}'(\tilde{S}) = 0$, then from Theorem 1 $\tilde{S}$ gives rise to the current algebra A'.

Conversely we have to prove that if $\tilde{S}(S)$ generates the current algebra A' by using the algebraic relations of S, then $\tilde{S}$ satisfies the equation $\mathbb{F}'(\tilde{S}) = 0$.

Due to the relation $F(S) = 0$, the S_i, $i = 1, 2, 3$, are not independent. Since $F(S) = 0$ is assumed to be a two-dimensional manifold, we can take, without losing generality, S_1, S_2 to be the independent variables. For $f, g \in \mathcal{F}(M_D)$, Theorem 2 says that the Poisson bracket of f and g is given by

$$[f, g]_{\mathrm{PB}} = \frac{1}{2} \int_D \frac{\delta \mathbb{F}(S)}{\delta S_3(\mathbf{y})} \left(\frac{\delta f}{\delta S_1(\mathbf{y})} - \frac{\delta f}{\delta S_2(\mathbf{y})} - \frac{\delta f}{\delta S_2(\mathbf{y})} \frac{\delta g}{\delta S_1(\mathbf{y})} \right) \mathrm{d}\mu(\mathbf{y}). \tag{36}$$

From Theorem 1 the Poisson current algebra A' is given by (11) with S replaced by $\tilde{S}$ and $\mathbb{F}$ by $\mathbb{F}'$. On the other hand from (36), the Poisson algebraic relations of $\tilde{S}_i(\mathbf{x})$ are

$$[\tilde{S}_i(\mathbf{x}), \tilde{S}_j(\mathbf{y})]_{\mathrm{PB}} = \frac{1}{2} \int_D \frac{\delta \mathbb{F}(S)}{\delta S_3(\mathbf{z})} \left(\frac{\delta \tilde{S}_i(\mathbf{x})}{\delta S_1(\mathbf{z})} \frac{\delta \tilde{S}_j(\mathbf{y})}{\delta S_2(\mathbf{z})} - \frac{\delta \tilde{S}_i(\mathbf{x})}{\delta S_2(\mathbf{z})} \frac{\delta \tilde{S}_j(\mathbf{y})}{\delta S_1(\mathbf{z})} \right) \mathrm{d}\mu(\mathbf{z}). \tag{37}$$

Taking into account that (as generalized functions)

$$\frac{\delta \tilde{S}_i(\mathbf{x})}{\delta S_j(\mathbf{y})} = \tilde{S}_{i,j}(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y})$$

(where $\tilde{S}_{i,j}(\mathbf{x})$ is the function of $\mathbf{x}$ obtained by evaluating the derivative of the function $\tilde{S}_i$ of S_j with respect to S_j at $\mathbf{x}$), we get from (11) (with the above replacements) and (37):

$$\frac{\delta \mathbb{F}(S)}{\delta S_3(\mathbf{y})} \left. \left(\frac{\delta \tilde{S}_i}{\delta S_1} \frac{\delta \tilde{S}_j}{\delta S_2} - \frac{\delta \tilde{S}_i}{\delta S_2} \frac{\delta \tilde{S}_j}{\delta S_1} \right) \right|_{\mathbf{y}} \delta(\mathbf{x} - \mathbf{y}) = \sum_{k=1}^{3} \epsilon_{ijk} \frac{\delta \mathbb{F}'(\tilde{S})}{\delta \tilde{S}_k(\mathbf{x})} \delta(\mathbf{x} - \mathbf{y})$$

By integrating with respect to $\mathrm{d}\mu(\mathbf{y})$ above equations for $i = 1$, $j = 2$ resp. $i = 2$, $j = 3$ resp. $i = 3$, $j = 1$ and multiplying the so-obtained equations by $\delta S'_3(\mathbf{x})/\delta S_k(\mathbf{z})$ resp. $\delta S'_1(\mathbf{x})/\delta S_k(\mathbf{z})$ resp. $\delta S'_2(\mathbf{x})/\delta S_k(\mathbf{z})$, summing then these equations together and finally integrating with respect to $\mathrm{d}\mu(\mathbf{x})$, we get

$$\frac{\delta \mathbb{F}'(\tilde{S})}{\delta S_k(\mathbf{z})} = 0, \quad k = 1, 2, \quad \forall \mathbf{z} \in D.$$

Therefore $\mathbb{F}'(\tilde{S})$ =constant. This is equivalent to the current manifold $\mathbb{F}'(\tilde{S}) = 0$ for the algebra A', since a constant term does not change the current algebras associated with the manifold. □

We give two examples of Poisson current algebraic maps. Eqs. (24) and (27) give algebraic maps between the current extended algebras of $SU(2)$ and $SU_q(2)$,

$$\begin{aligned} S'_{\pm}(\mathbf{x}) &= \frac{1}{\sqrt{\gamma \sinh \gamma}} S_{\pm}(\mathbf{x}) \frac{\sinh \gamma (S_0 \mp S_3(\mathbf{x}))}{S_0 \mp S_3(\mathbf{x})}, \\ S'_3(\mathbf{x}) &= S_3(\mathbf{x}), \end{aligned} \tag{38}$$

where $S_{\pm}(\mathbf{x}) = S_1(\mathbf{x}) \pm \mathrm{i} S_2(\mathbf{x})$. It is easy to check by using the relations (25) that $S'_{\pm}(\mathbf{x})$, $S'_3(\mathbf{x})$ satisfy (28). They also satisfy (27) as seen by using (24).

The maps relating the generators of the current algebras of $\mathcal{H}(4)$ to those of $\mathcal{H}_q(4)$ can be obtained from the related manifolds (30) and (33). For instance,

$$\begin{aligned} S'_{+}(\mathbf{x}) &= \frac{\sinh \gamma (S_3(\mathbf{x}) + 1/2)}{(S_3(\mathbf{x}) + 1/2)\gamma} S_{+}(\mathbf{x}), \\ S'_{-}(\mathbf{x}) &= \frac{\cosh \gamma (S_3(\mathbf{x}) - 1/2)}{\cosh \gamma} S_{-}(\mathbf{x}), \\ S'_3(\mathbf{x}) &= S_3(\mathbf{x}), \end{aligned} \tag{39}$$

where $S_{\pm}(\mathbf{x})$ and $S_3(\mathbf{x})$ are the generators of the current algebra $\mathcal{H}(4)$ satisfying relations (31). $S'_{\pm}(\mathbf{x})$ and $S'_3(\mathbf{x})$ are the generators of the current algebra $\mathcal{H}_q(4)$ satisfying relations (34). We also see that $S_{\pm}(\mathbf{x})$, $S_3(\mathbf{x})$ satisfy (30) and $S'_{\pm}(\mathbf{x})$, $S'_3(\mathbf{x})$ satisfy (33).

Similarly, the corresponding maps for other current extended algebras such as $SU(1,1)$ and $SU_q(1,1)$ can be studied by investigating their related current manifolds.

6. Geometric quantization of current algebras

Towards geometric quantization one has to construct a linear monomorphism from the Poisson algebra of (M_N, ω) to the space of linear operators on an appropriate space by constructing the prequantization line bundle L and introduce a suitable polarization P (cf. [11,12]).

In the following we take the current extended algebra of $SU(2)$ as an example. As is shown in Section 4 this algebra is related to the current 2D sphere defined by Eq. (24). Let S^2 denote the latter manifold. We set up a complex structure on S^2 by introducing two open sets $U_{\pm} = \{z \in S^2 | S_0 \pm S_3(z) \neq 0\}$ and two complex functions $z_+(\mathbf{x})$ and $z_-(\mathbf{x})$, $\mathbf{x} \in N$, on U_+ and U_-, respectively,

$$z_{\pm}(\mathbf{x}) = \frac{S_1(\mathbf{x}) \mp \mathrm{i} S_2(\mathbf{x})}{S_0 \pm S_3(\mathbf{x})}. \tag{40}$$

In $U_+ \cap U_-$ we have $z_+(\mathbf{x}) z_-(\mathbf{x}) = 1$.

From (40) we get the expressions of $S_i(\mathbf{x})$, $i = 1, 2, 3$, in terms of $z_+(\mathbf{x})$ and $z_-(\mathbf{x})$,

$$S_1(\mathbf{x}) = S_0 \frac{z_\pm(\mathbf{x}) + \bar{z}_\pm(\mathbf{x})}{1 + z_\pm(\mathbf{x})\bar{z}_\pm(\mathbf{x})},$$
$$S_2(\mathbf{x}) = \pm \mathrm{i} S_0 \frac{z_\pm(\mathbf{x}) - \bar{z}_\pm(\mathbf{x})}{1 + z_\pm(\mathbf{x})\bar{z}_\pm(\mathbf{x})}, \tag{41}$$
$$S_3(\mathbf{x}) = \pm S_0 \frac{1 - z_\pm(\mathbf{x})\bar{z}_\pm(\mathbf{x})}{1 + \mathbf{z}_\pm(\mathbf{x})\bar{z}_\pm(\mathbf{x})}.$$

The symplectic form (26) becomes

$$\omega | U_\pm = \int_N \frac{-2\mathrm{i} S_0}{(1 + z_\pm(\mathbf{x})\bar{z}_\pm(\mathbf{x}))^2} \delta\bar{z}_\pm(\mathbf{x}) \wedge \delta z_\pm(\mathbf{x})\, \mathrm{d}\mu(\mathbf{x}). \tag{42}$$

There is a locally defined one form θ s.t. defining $\theta_\pm = \theta | U_\pm$ one has $\delta\theta_\pm = \omega | U_\pm$ and

$$\theta_\pm = \int_N \frac{-2\mathrm{i} S_0}{1 + z_\pm(\mathbf{x})\bar{z}_\pm(\mathbf{x})} \bar{z}_\pm(\mathbf{x}) \delta z_\pm(\mathbf{x})\, \mathrm{d}\mu(\mathbf{x}). \tag{43}$$

The Hamiltonian vector fields of $S_i(\mathbf{x})$ now take the form

$$X_{S_{1(\mathbf{x})}} | U_\pm = -\frac{\mathrm{i}}{2} \left[(z_\pm^2(\mathbf{x}) - 1) \frac{\delta}{\delta z_\pm(\mathbf{x})} + (1 - \bar{z}_\pm^2(\mathbf{x})) \frac{\delta}{\delta \bar{z}_\pm(\mathbf{x})} \right],$$
$$X_{S_{2(\mathbf{x})}} | U_\pm = \pm\frac{1}{2} \left[(z_\pm^2(\mathbf{x}) + 1) \frac{\delta}{\delta z_\pm(\mathbf{x})} + (1 + \bar{z}_\pm^2(\mathbf{x})) \frac{\delta}{\delta \bar{z}_\pm(\mathbf{x})} \right], \tag{44}$$
$$X_{S_{3(\mathbf{x})}} | U_\pm = \pm\mathrm{i} \left(\bar{z}_\pm(\mathbf{x}) \frac{\delta}{\delta \bar{z}_\pm(\mathbf{x})} - z_\pm(\mathbf{x}) \frac{\delta}{\delta z_\pm(\mathbf{x})} \right).$$

We suppose that the manifold N has finite volume V. The prequantization line bundle L exists if and only if the de Rham cohomology class $[-V^{-1}h^{-1}\omega]$ of $-V^{-1}h^{-1}\omega$ is integrable [12]. This leads to the relation

$$-V^{-1}h^{-1} \int_{S^2} \omega = 2j, \quad 2j \in \mathbb{N}.$$

Hence we have $S_0 = j\hbar$. In the following discussions we shall, for simplicity, set $\hbar = 1$.

We take, as a suitable polarization, the linear frame fields of $z_\pm(\mathbf{x})$,

$$X_{z_{\pm(\mathbf{x})}} = (2\mathrm{i} S_0)^{-1} (1 + z_\pm(\mathbf{x})\bar{z}_\pm(\mathbf{x}))^2 \frac{\delta}{\delta \bar{z}_\pm(\mathbf{x})}.$$

On $U_+ \cap U_-$, $z_+(\mathbf{x}) \neq 0$, we have $X_{z_{-(\mathbf{x})}} = -z_+^{-2}(\mathbf{x}) X_{z_{+(\mathbf{x})}}$. Hence, $X_{z_{+(\mathbf{x})}}$ and $X_{z_{-(\mathbf{x})}}$ span a complex distribution P on S^2 and P is a polarization of the symplectic manifold (S^2, ω). Moreover,

$$\mathrm{i}\omega(X_{z_{\pm(\mathbf{x})}}, \bar{X}_{z_{\pm(\mathbf{x})}}) = \frac{1}{2S_0} (1 + z_\pm(\mathbf{x})\bar{z}_\pm(\mathbf{x}))^2 > 0.$$

This means that (adapting a definition from [12]) P is a complete strongly admissible positive polarization of (S^2, ω) (in the sense of generalized functions).

From Eqs. (41) and (44) we obtain, applying methods analogous to the ones of finite-dimensional geometric quantization, the following definition of the corresponding quantum operators:

$$\begin{aligned}
\hat{S}_1(\mathbf{x})|U_\pm &\equiv -\frac{1}{2}(z_\pm^2(\mathbf{x})-1)\frac{\delta}{\delta z_\pm(\mathbf{x})}+jz_\pm(\mathbf{x}),\\
\hat{S}_2(\mathbf{x})|U_\pm &\equiv \mp\frac{\mathrm{i}}{2}(z_\pm^2(\mathbf{x})+1)\frac{\delta}{\delta z_\pm(\mathbf{x})}\pm \mathrm{i}jz_\pm(\mathbf{x}),\\
\hat{S}_3(\mathbf{x})|U_\pm &\equiv \pm\left(-z_\pm(\mathbf{x})\frac{\delta}{\delta z_\pm(\mathbf{x})}+j\right),
\end{aligned}\tag{45}$$

as operators acting on some space $\mathcal{F}$ of functionals of $z_\pm$. They give rise to the quantum current extended algebra of $SU(2)$,

$$\begin{aligned}
[\hat{S}_+(\mathbf{x}), \hat{S}_-(\mathbf{y})] &= 2\hat{S}_3(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}),\\
[\hat{S}_3(\mathbf{x}), \hat{S}_\pm(\mathbf{y})] &= \pm\hat{S}_\pm(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}),
\end{aligned}\tag{46}$$

where $\hat{S}_\pm(\mathbf{x}) = \hat{S}_1(\mathbf{x}) \pm \mathrm{i}\hat{S}_2(\mathbf{x})$ ($[A, B]$ denotes $AB - BA$). These relations have to be understood in the sense of operator-valued generalized functions, the operators acting in $\mathcal{F}$.

To pass to a q-version it is useful to discretize the N-manifold (avoiding in this way some divergences which arise in the following computations in the continuous case). We thus replace N by a discretized version of it, N_d. In this case we get from (45) for any $\mathbf{x} \in N_d$,

$$(j+\tfrac{1}{2})^2 = \hat{S}_+(\mathbf{x})\hat{S}_-(\mathbf{x}) + (\hat{S}_3(\mathbf{x})-\tfrac{1}{2})^2 \tag{47}$$

This relation gives a "quantum mechanical" analogue of the sphere S^2.

The geometric quantization of the current algebra of $SU_\mathrm{q}(2)$ can be studied in a similar-way. Let S_q^2 denote the manifold defined by Eq. (27). Then the prequantization line bundle on (S_q^2, ω) exists iff when $-V^{-1}h^{-1}\int_{S_\mathrm{q}^2}\omega = 2j$, $2j \in \mathbb{N}$, where ω is given by formula (29). This leads to an S_γ in (27) which takes the form (for $\gamma \neq 0$)

$$S_\gamma = \frac{\sinh\gamma S_0}{\sqrt{\gamma\sinh\gamma}} = \frac{\sinh\gamma j}{\sqrt{\gamma\sinh\gamma}}.$$

By geometric quantization (in a sense similar as above) we have the quantum operators of the current extended algebra of $SU_\mathrm{q}(2)$:

$$\begin{aligned}
\hat{S}_1(\mathbf{x})|V_\pm &= \frac{1}{\sqrt{\gamma\sinh\gamma}}(A_\pm + B_\pm),\\
\hat{S}_2(\mathbf{x})|V_\pm &= \frac{\pm\mathrm{i}}{\sqrt{\gamma\sinh\gamma}}(A_\pm - B_\pm),
\end{aligned}\tag{48}$$

$$\hat{S}_3(\mathbf{x})|V_\pm = \pm\left(-z_\pm(\mathbf{x})\frac{\delta}{\delta z_\pm(\mathbf{x})} + j\right),$$

where

$$A_\pm \equiv \cosh\left(\frac{\gamma}{2} z_\pm(\mathbf{x})\frac{\delta}{\delta z_\pm(\mathbf{x})}\right) z_\pm(\mathbf{x}) \sinh\left(\frac{\gamma}{2}\left(-z_\pm(\mathbf{x})\frac{\delta}{\delta z_\pm(\mathbf{x})} + 2j\right)\right),$$
$$B_\pm \equiv \cosh\left(\frac{\gamma}{2}(-z_\pm(\mathbf{x})\frac{\delta}{\delta z_\pm(\mathbf{x})} + 2j\right)\frac{1}{z_\pm(\mathbf{x})} \sinh\left(\frac{\gamma}{2} z_\pm(\mathbf{x})\frac{\delta}{\delta z_\pm(\mathbf{x})}\right)$$

(in the sense of operator-valued generalized functions), where the open sets $V_\pm$ on S_q^2 is defined by

$$V_\pm = \left\{z \in s_q^2 | S_\gamma \pm \frac{\sinh \gamma S_3(z)}{\sqrt{\gamma \sinh \gamma}} \neq 0\right\}.$$

They satisfy (in a similar sense as above) the commutation relations of the current extended quantum algebra of SU_q (2),

$$[\hat{S}_+(\mathbf{x}), \hat{S}_-(\mathbf{y})] = \frac{\sinh(\gamma)}{\gamma}\frac{\sinh 2\gamma \hat{S}_3(\mathbf{x})}{\sinh \gamma}\delta(\mathbf{x}-\mathbf{y}),$$
$$[\hat{S}_3(\mathbf{x}), \hat{S}_\pm(\mathbf{y})] = \pm \hat{S}_\pm(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}), \tag{49}$$

where $\hat{S}_\pm(\mathbf{x}) = \hat{S}_1(\mathbf{x}) \pm \mathrm{i}\hat{S}_2(\mathbf{x})$, $x, y \in N_d$. This quantum current algebra is isomorphic to the classical Poisson current algebra (28). One may redefine $\hat{S}_\pm(\mathbf{x})$ by multiplying it by a constant factor $\sqrt{\sinh(\gamma)/\gamma}$ so as to get the usual form of commutation relations.

In terms of expressions (48) and (49) the quantum operators satisfy the following equation:

$$\hat{S}_+(\mathbf{x})\hat{S}_-(\mathbf{x}) + \frac{\sinh^2 \gamma(\hat{S}_3(\mathbf{x}) - 1/2)}{\gamma \sinh \gamma} = \frac{\sinh^2 \gamma(j + 1/2)}{\gamma \sinh \gamma}, \tag{50}$$

This is the quantum version of the manifold (27).

The quantization of other current algebras can be discussed in a similar way. In addition, the quantum current algebraic maps can be obtained from the quantum version of the related manifolds. For instance, let $\hat{S}_i(\mathbf{x})$ (resp. $\hat{S}'_i(\mathbf{x})$) be the quantum operators of the quantum current algebra of $SU(2)$ (resp. $SU_q(2)$) satisfying Eq. (47) (resp. (50)). We set $\hat{S}'_3(\mathbf{x}) = \hat{S}_3(\mathbf{x})$. Then Eq. (50) can be rewritten as

$$\begin{aligned}\hat{S}'_+(\mathbf{x})\hat{S}'_-(\mathbf{x}) &= \frac{\sinh^2 \gamma(j + 1/2)}{\gamma \sinh \gamma} - \frac{\sinh^2 \gamma(\hat{S}_3(\mathbf{x}) - 1/2)}{\gamma \sinh \gamma} \\ &= \frac{1}{\sqrt{\gamma \sinh \gamma}}\hat{S}_+(\mathbf{x})\frac{\sinh \gamma(j - \hat{S}_3(\mathbf{x}))}{j - \hat{S}_3(\mathbf{x})} \\ &\quad\times \frac{1}{\sqrt{\gamma \sinh \gamma}}\hat{S}_-(\mathbf{x})\frac{\sinh \gamma(j + \hat{S}_3(\mathbf{x}))}{j + \hat{S}_3(\mathbf{x})},\end{aligned}$$

where Eq. (47) has been used. Hence we have

$$\hat{S}'_\pm(\mathbf{x}) = \frac{1}{\sqrt{\gamma \sinh \gamma}}\hat{S}_\pm(\mathbf{x})\frac{\sinh \gamma(j \mp \hat{S}_3(\mathbf{x}))}{j \mp \hat{S}_3(\mathbf{x})}, \quad \hat{S}'_3(\mathbf{x}) = \hat{S}_3(\mathbf{x}). \tag{51}$$

Therefore from the "quantum" current manifold (47) of the current extended quantum algebra $SU(2)$ and the "quantum" current manifold (50) of the current extended quantum algebra $SU_q(2)$, we get the quantum algebraic maps from the current extended algebras $SU(2)$ to $SU_q(2)$, which are formally the same as the classical Poisson algebraic maps (38). Here $\hat{S}'_{3,\pm}(\mathbf{x})$ satisfy relations (50) while $\hat{S}_{3,\pm}(\mathbf{x})$ satisfy (47), as guaranteed by Theorem 3.

7. Conclusion and remarks

We have shown that there is a one-to-one correspondence (up to algebraic equivalence) between Poisson current algebras of maps from a (Riemannian) manifold N to Poisson algebras with three generators and current manifolds M_N (with dim $M = 2$). This gives rise to a general description of such Poisson current algebras resp. their quantum versions. A geometric meaning of q-deformation of such current algebras emerges in terms of the corresponding q-deformation of the current manifolds resp. their quantum versions. Maps between two Poisson current algebras (resp. quantum current algebras) can be simply handled in terms of their associated current manifolds (resp. the quantum versions of these current manifolds).

The Hopf structures (for this concept see e.g. [19]) of current extended algebras can be studied by the quantum operators of these algebras. Here we give the Hopf structure for the current extended algebra of $SU_q(2)$:

$$\begin{aligned}
&\Delta(\hat{S}_3(\mathbf{x})) = \hat{S}_3(\mathbf{x}) \otimes \mathbf{1} + \mathbf{1} \otimes \hat{S}_3(\mathbf{x}),\\
&\Delta(\hat{S}_\pm(\mathbf{x})) = \hat{S}_\pm(\mathbf{x}) \otimes \mathrm{e}^{-\gamma \hat{S}_3(\mathbf{x})} + \mathrm{e}^{-\gamma \hat{S}_3(\mathbf{x})} \otimes \hat{S}_\pm(\mathbf{x}),\\
&\varepsilon(\mathbf{1}) = \mathbf{1}, \qquad \varepsilon(\hat{S}_\pm(\mathbf{x})) = \varepsilon(\hat{S}_3(\mathbf{x})) = 0,\\
&\eta(\hat{S}_\pm(\mathbf{x}) = -\mathrm{e}^{\gamma \hat{S}_3(\mathbf{x})} \hat{S}_\pm(\mathbf{x}) \mathrm{e}^{-\gamma \hat{S}_3(\mathbf{x})}, \qquad \eta(\hat{S}_3(\mathbf{x})) = -\hat{S}_3(\mathbf{x}),
\end{aligned} \tag{52}$$

where Δ, η and ε are coproduct, antipode and counit operations, respectively. These operations conserve the quantum current algebraic relations (49). When the deformation parameter γ approaches zero, (52) becomes formally the Hopf algebraic structures of the quantum current extended algebra of $SU(2)$ and the comultiplication becomes commutative.

In this paper we have obtained the quantum operators associated with the current extended algebras through geometric quantization. In particular we have constructed here a representation of the current extended quantum algebra $SU_q(2)$, see (49). Representations of the current extended quantum algebras can also be obtained in the form of highest weight representations based on continuous tensor product representations [20]. It would be interesting to study the relations between the latter representations and the representations obtained in this paper. In addition, by generalizing the 2D current manifolds to Grassmannian manifolds, one can discuss BRST structures on 2D current manifolds in terms of symplectic geometry and geometric quantization, extending the work we have done before for 2D manifolds [21].

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